Environmetrics; 1992; 3(3): 301-318

A Simple Protocol for Radiation Dosimetry Based on the Order Statistics from a Poisson Process

David R. Fox[1]

ABSTRACT

In this paper problems associated with the sampling and estimation of the background gamma radiation dose-rate are considered. The work described herein has been motivated by the large sampling effort of background gamma radiation undertaken in Western Australia by ALCOA of Australia coupled with the absence of any definitive sampling methodology. Certain best estimators are proposed and results given which permit the construction of 'optimal' sampling protocols for radiological investigations.

1. INTRODUCTION

Much has been written and published about the many aspects of radiation monitoring and evaluation [see for example Mann *et al.* (1980) and Tsoulfanidis (1983)] although surprisingly little or no attention has been given to the mechanics of the measurement process itself.

Chapter five of NCRP Report #57 (1978) is devoted to instrumentation methods for radiation protection. This otherwise comprehensive treatment of the subject discusses types of instruments and their uses, calibration procedures, environmental and personnel monitoring but gives no indication of how measurements should be obtained. Similarly, NCRP Report #50 (1985a, pp 75–76) details a number of desirable features for portable scintillation and Geiger-Muller (G-M) instruments — none of which relate to the nature of information to be recorded, the method of data collection, or estimation procedures. The report also lists (p 171) a number of sources

[1] Department of Mathematics, University of Colorado at Denver.

1180-4009/92/030301-18$14.00

Received 6 March 1992
Revised 15 May 1992

of errors and detection limits associated with radioactivity measurements, again, none of which arises from the sampling protocol or the estimators used.

The somewhat parlous state of sampling methodology for radiation dosimetry and the apparent statistical naivety of many researchers is conveyed by Ackers (1978):

> Part of the task of a secondary standard dosimetry laboratory is to calibrate low-level gamma radiation monitors for field use. Many of these instruments are portable, battery-operated, exposure-rate meters with readings taken from a pointer indication on a scale calibrated in μR/h to mR/h. Using such an instrument, the observer reads a "mean value", determined from the oscillations of a moving pointer. His reading has a statistical reliability, the numerical value of which many observers are not completely aware of. If a guess is made concerning the accuracy of the estimated reading, the observer often has no correct information on which to base his findings.

That the author placed quotes around *mean value* indicates his own uncertainty in what is actually meant. It is left to the practitioner as to how this *mean value* is obtained "from the oscillations of a moving pointer". The statistical double-speak of the last two sentences is confusing to say the least. Ackers clearly means *variability* instead of reliability, although the sentence still does not convey what is really intended about the standard error of the estimator of the true mean. The suggestion that a *guess* be made for this quantity truly reflects the inadequacies of radiation sampling methodology. In a subsequent paragraph (p 528), the author nevertheless acknowledges the importance of this variability:

> The statistical error of a reading is, however, important if the readings are to be used as a basis for decision making, eg. in radioactive waste management — when judgment for permission to transport has to be made.

It is for precisely this reason that the present research was undertaken as part of a much broader project associated with alternative uses of alumina refinery residues. ALCOA of Australia undertook a large survey to quantify the radiological profile of the naturally occuring background gamma radiation in the Swan Coastal Plain of Western Australia. For various reasons it was decided to use ground-based sampling with hand-held instruments in preference to an air-borne survey. The instrument used was a Geiger-Muller meter that has an analog (needle) display which fluctuates in accordance with the number of radioactive particles reaching the detector. An extensive literature search and discussions with professionals working in radiation laboratories has confirmed the lack of definitive advice regarding the statistical aspects of the measurement process. The only suggestion we have uncovered is again due to Ackers (p 528):

> A method of calculating the mean value and the standard deviation involves taking a sample of the continuous indication by reading every five seconds, exactly, the position of the oscillating pointer, regardless of the direction and speed of its movement. By doing this for about three minutes, a reliable sample (n observations) of the statistically distributed positions of the pointer is obtained.

It is not clear from the above whether it is the time interval or meter position that is to be measured *exactly*. Either way, the above suggestion serves no useful purpose and can only be viewed as a utopian goal rather than an implementable regime for measurement.

As noted by McLaughlin (1978, p 487), "there appears to be a need to improve the quality of environmental radiation measurements ... efforts are needed on instrument development and determining instrument capabilities ie. sensitivity, and reliability, and calibration".

In terms of the present study, two fundamental questions required answers:

(i) what sampling period should be used ?

and

(ii) what information from the meter should be recorded? (for example, should the operator try to 'eyeball' an 'average' reading?; should the largest and smallest readings be recorded?; or should some other measure be used?).

The difficulties associated with taking readings from a sometimes rapidly fluctuating meter are obvious. Another common approach is to simply average the smallest and largest ($Y_{\min}$ and $Y_{\max}$ respectively) readings observed during the recording period. While having some intuitive appeal and ease of recording, the sampling properties of this estimator needs to be examined if meaningful inference is to be made on the process under investigation. At this stage of the radiological survey, question (i) above remained unanswered. With respect to question (ii) there seemed to be few choices and it was felt that the observation of the extreme readings was less prone to error than the subjective determination of some kind of 'average' reading.

The remainder of this paper is devoted to the statistical assessment of properties of the class of estimators formed by taking linear combinations of $Y_{\min}$ and $Y_{\max}$ under the assumption that the phenomenon being observed represents a Poisson process. First, coefficients are provided which lead to the unbiased estimation of the true mean dose-rate. Using these coefficients, the standard error of the estimate is computed for a range of sampling periods. Examination of the resulting plots enables both the 'best' weighting of $Y_{\min}$ and $Y_{\max}$ and the sampling interval to be determined over a wide range of actual dose-rates. In addition, seemingly unrelated regression (SUR) models and estimated generalized least-squares (EGLS) procedures have been used to establish calibration equations relating true mean dose-rate to the expected value of the sample order statistics for a variety of sampling intervals. Future use of these equations permits inference to be made on the true mean dose-rate on the basis of observed minimum and maximum meter readings.

2. Characteristics of Hand-Held Instruments

The choice of a measuring system for a particular application is usually dictated by considerations of cost, time constraints, instrument efficiency, and the type and intensity of the radiation to be measured. Portable instruments offer an inexpensive and convenient method of measuring radiation dose-rates. Geiger-Muller counters are used for the detection and measurement of x- and gamma-ray fields and since this type of instrument responds to the *number* of ionizing events rather than the energy associated with the events, these instruments should only be used for measuring low-level dose-rates (NCRP 1978).

Two limiting features of G-M meters are the *dead time* and the *time constant*. Dead time is defined as the minimum time interval by which two consecutive events must be separated for both to be recorded. Events arriving during this dead time are lost and the fraction of lost events increases with increasing counting rate (NCRP 1985b). For measurements of natural background gamma radiation, dead time will be unimportant. On the other hand, an instrument's *time constant* is central to the determination of an appropriate sampling interval. The time constant reflects the instrument's inertia or lack of responsiveness to a changing field. Time constants are sometimes reported as the time required to arrive at 63% $[(1 - e^{-1})100\%]$ of the final reading. G-M meters have typical response times of about *one second* (NCRP 1978, p 95).

3. Modelling the Gamma Dose-Rate

3.1 Units of Measurement

Various units of measurement are used in conjunction with radiation dosimetry although in this paper we shall restrict ourselves to *rads*. An exposure of one roentgen is frequently considered approximately equivalent to an absorbed dose of one rad, and the unit roentgen is loosely (although incorrectly) used to mean rad (Cember 1983).

The following statistical development is applicable to *any* hand-held instrument, although for the the results to be implementable the conversion factor relating *events per second* and *dose rate* needs to be established. As an example, using energy absorption coefficients of 3.46×10^{-5} cm^{-1} for air at 20°C and 0.0312 cm^{-1} for tissue for 0.3 MeV photons and a tissue density of 0.0001 kg/cm the absorbed dose-rate relationship is found to be:

$$1 \text{ event/second} \equiv 1.5 \times 10^{-10} \text{ Rad/sec.}$$

or equivalently

$$100\ \mu\text{Rad/hour} \equiv 185 \text{ events/second}$$

where an "event" is defined to be the registration of a gamma particle on the G-M tube.

3.2 METHOD

Our notion of a random process evolving over time is conceptualized in Figure 1. The random variable Z denotes the number of gamma particle registrations or counts per unit time. It is assumed that Z follows a Poisson process having mean rate λ.

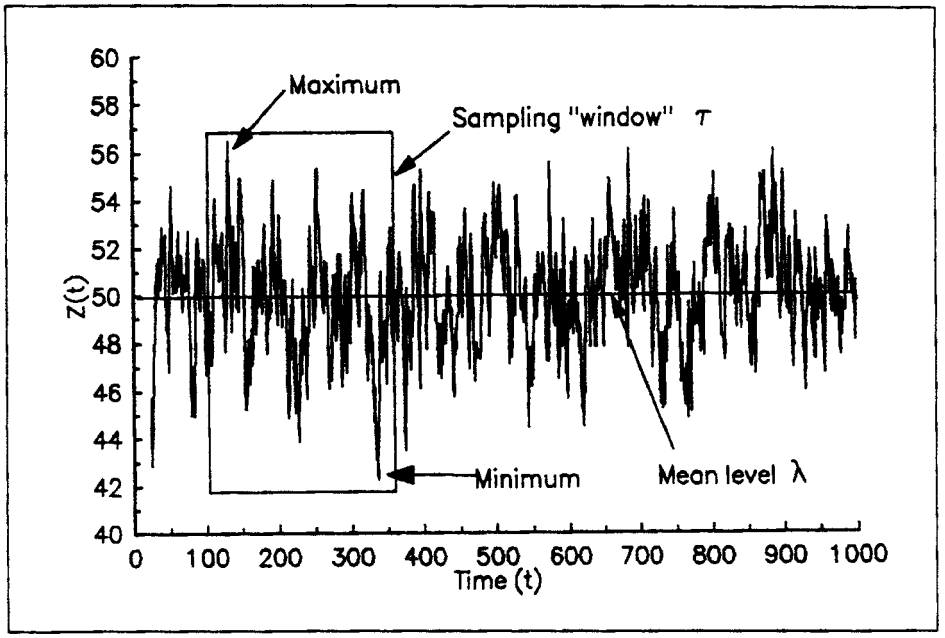

Figure 1. Random process plotted over time.

Our problem is two-fold: For given λ we wish to determine an 'optimal' sampling interval τ and secondly, determine appropriate weights for $Y_{\min}$ and $Y_{\max}$ which result in the unbiased estimation of λ. The criterion of minimum variance will be used to quantify 'optimal'.

Let the sampling interval τ be divided into k sub-intervals of equal length Δt, where Δt is the time constant for the particular instrument. Thus $\tau = k\Delta t$. The apparent continuous indication of the meter's display is thus represented by the discrete observations $X_1, X_2, \ldots, X_k$. Furthermore let the associated order statistics be denoted $Y_1, Y_2, \ldots, Y_k$. Our estimator of λ is assumed to be of the form:

$$\hat{\lambda} = \alpha Y_1 + (1 - \alpha) Y_k \quad . \tag{3.2.1}$$

The condition of unbiased estimation imposes the following restriction:

$$E(\hat{\lambda}) = \lambda \Rightarrow \alpha E(Y_1) + (1 - \alpha) E(Y_k) = \lambda \tag{3.2.2}$$

where E is the expectation operator.

Thus, once $\mu_1 = E(Y_1)$ and $\mu_k = E(Y_k)$ are known for given λ and k, we have:

$$\alpha = \frac{\mu_k - \lambda}{\mu_k - \mu_1} \quad . \tag{3.2.3}$$

Furthermore, for any α the variance of $\hat{\lambda}$ is

$$\sigma^2(\hat{\lambda}) = \mathrm{Var}(\hat{\lambda}) = \alpha^2\sigma_{11} + 2\alpha(1-\alpha)\sigma_{1k} + (1-\alpha)^2\sigma_{kk} \tag{3.2.4}$$

where σ_{11} and σ_{kk} are respectively the variances of Y_1 and Y_k and σ_{1k} is the covariance between them. For given λ and k we can evaluate μ_1, μ_k, σ_{11}, σ_{kk} and σ_{1k}. Eq. (3.2.3) is then used to determine α and this in turn is used in Eq. (3.2.4) to evaluate $\sigma^2(\hat{\lambda})$. By repeating this sequence of steps for various combinations of λ and k we can establish plots of $\sigma(\hat{\lambda})$ versus λ — there being one plot for each k. Assuming that we have some idea of the magnitude of the background gamma dose-rate (this is typically 5–20 μRad/hour) we can use the plots to determine values of τ and α that minimize $\sigma^2(\hat{\lambda})$. Clearly this procedure is dependent upon the evaluation of the population parameters μ_1, μ_k, σ_{11}, σ_{kk}, and σ_{1k} for given λ and k. The derivation and computation of these quantities is considered in the following section.

3.3 Population Moments for Y_1 and Y_k

Under the assumption of a Poisson process, the distribution of X_i is

$$f_{X_i}(x_i) = \frac{e^{-\lambda\Delta t}(\lambda\Delta t)^{x_i}}{x_i!} \quad ; \qquad x_i = 0,1,2,\ldots \qquad \lambda > 0 \quad . \tag{3.3.1}$$

For the time being we will assume $\Delta t = 1$ and later generalize our results to any Δt. Since the X_i are i.i.d. random variables we have

$$f_{X_i}(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; \qquad x = 0,1,2,\ldots \qquad \lambda > 0 \tag{3.3.2}$$

and associated c.d.f

$$F_X(t) = \sum_{x=0}^{t} f_X(x) \quad . \tag{3.3.3}$$

For a *discrete* random variable, the probability functions for the minimum and maximum from a sample of size k are, respectively:

$$f_{Y_1}(y_1) = \{1 - F_X(y_1 - 1)\}^k - \{1 - F_X(y_1)\}^k \tag{3.3.4}$$

and

$$f_{Y_k}(y_k) = [F_X(y_k)]^k - \{F_X(y_k - 1)\}^k \quad . \tag{3.3.5}$$

The following joint probability function for any two order statistics Y_i and Y_j, $i < j$ from a discrete distribution is given by David (1981):

$$f_{Y_i,Y_j}(y_i, y_j) =$$
$$\sum_{l=0}^{i-1}\sum_{m=0}^{k-j}\sum_{u,t}\frac{k!}{(i-l-1)!(1+l+t)!(j-i-1-u-t)!(1+m+u)!(k-j-m)!}$$
$$\times\{F_X(y_1-1)\}^{i-l-1}\{f_X(y_1)\}^{1+l+t}\{F_X(y_k-1)-F_X(y_1)\}^{j-i-1-u-t}$$
$$\times\{F_X(y_k)\}^{1+m+u}\{1-F_X(y_k)\}^{k-j-m} \tag{3.3.6}$$

where $\sum_{u,t}$ denotes summation over non-negative integral values of u and t subject to $u+t \le j-i-1$. Setting $i=1$ and $j=k$ we have:

$$f_{Y_1,Y_k}(y_1, y_k) = \sum_{u,t}\frac{k!}{(t+1)!(u+1)!(k-u-t-2)!}\{f_X(y_1)\}^{t+1}$$
$$\times\{F_X(y_k-1)-F_X(y_1)\}^{k-u-t+2}\{F_X(y_k)\}^{u+1} \tag{3.3.7}$$

with $u+t \le k-2$.

A computationally more efficient version of (3.3.6) is also given by David (1981, p 14, Eq. 2.4.4) and for $i=1$ and $j=k$ this reduces to:

$$f_{Y_1,Y_k}(y_1, y_k) = k(k-1)\int\int(v-w)^{k-2}\,dv\,dw \tag{3.3.8}$$
$$w \le v$$
$$F_X(y_1-1) \le w \le F_X(y_1)$$
$$F_X(y_k-1) \le v \le F_X(y_k)$$

Using Eqs. (3.3.4), (3.3.5), and (3.3.8) the following population moments may be computed:

$$\mu_1 = E(Y_1) = \sum_{y_1=0}^{\infty} y_1 f_{Y_1}(y_1) \tag{3.3.9}$$

$$\mu_k = E(Y_k) = \sum_{y_k=0}^{\infty} y_k f_{Y_k}(y_k) \tag{3.3.10}$$

$$\sigma_{11} = \mathrm{Var}(Y_1) = \sum_{y_1=0}^{\infty} y_1^2 f_{Y_1}(y_1) - \mu_1^2 \tag{3.3.11}$$

$$\sigma_{kk} = \mathrm{Var}(Y_k) = \sum_{y_k=0}^{\infty} y_k^2 f_{Y_k}(y_k) - \mu_k^2 \tag{3.3.12}$$

$$\sigma_{1k} = \mathrm{Cov}(Y_1, Y_2) = \sum_{y_1=0}^{\infty}\sum_{y_k=y_1}^{\infty} y_1 y_k f_{Y_1,Y_k}(y_1, y_k) - \mu_1\mu_k \tag{3.3.13}$$

3.4 Results

Equations (3.3.9)–(3.3.13) together with Eqs. (3.2.3) and (3.2.4) were programmed using the matrix-based language GAUSS and run concurrently on several 386-PC computers equipped with math co-processors. A plot of the results displaying the relationship between the weighting factor α and the true mean dose-rate, λ for various sampling intervals is given in Figure 2.

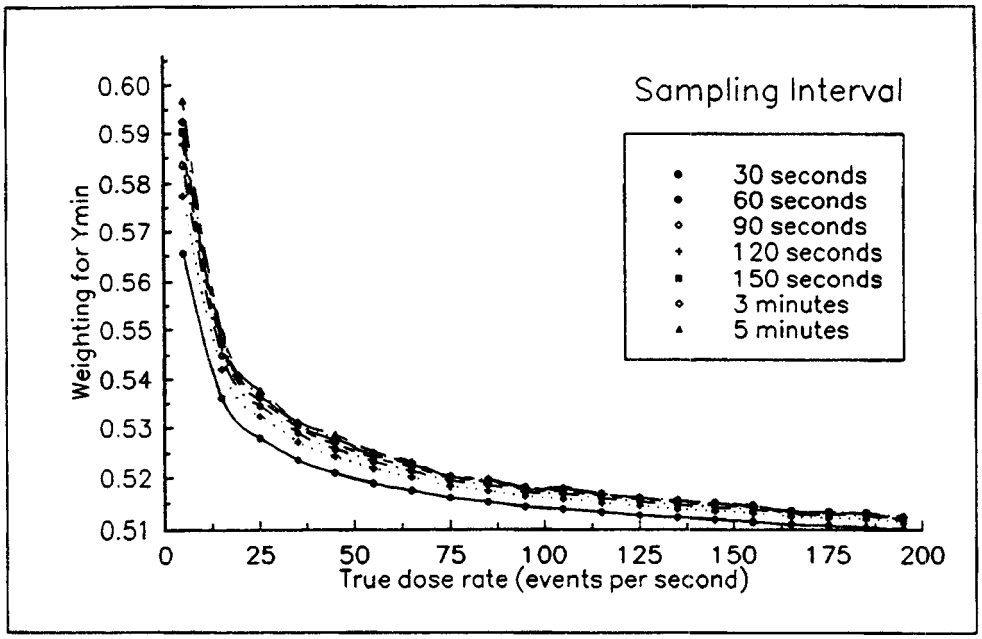

Figure 2. Plot of α versus true mean dose-rate (λ).

As can be seen from Figure 2, all seven curves display the same basic inverse relation between the weighting factor alpha and the true mean dose-rate. It would appear that the curve converges to about 0.5 for high dose-rates suggesting that at higher radiation levels the simple average of $Y_{\mathbf{min}}$ and $Y_{\mathbf{max}}$ provides an unbiased estimator of the true mean dose-rate. Another general observation is that for any given dose-rate, $Y_{\mathbf{min}}$ is weighted more heavily for long sampling intervals than for shorter ones. Note also that the weighting for $Y_{\mathbf{min}}$ is always greater than for $Y_{\mathbf{max}}$.

Using the values of α suggested by Figure 2 together with Eq. (2.2.4), we can compute the standard error of the estimator $\hat{\lambda}$ as a function of the true mean dose-rate for each sampling interval. This has been done and the results displayed in Figure 3. Figure 4 shows the same plot on a log-log scale. The strong linearity displayed by the latter plot suggests that the standard error of $\hat{\lambda}$ is a power function of the true mean dose-rate. A regression analysis of the log-transformed data revealed that the lines in Figure 4 are significantly different from each other ($p = 0.0001$).

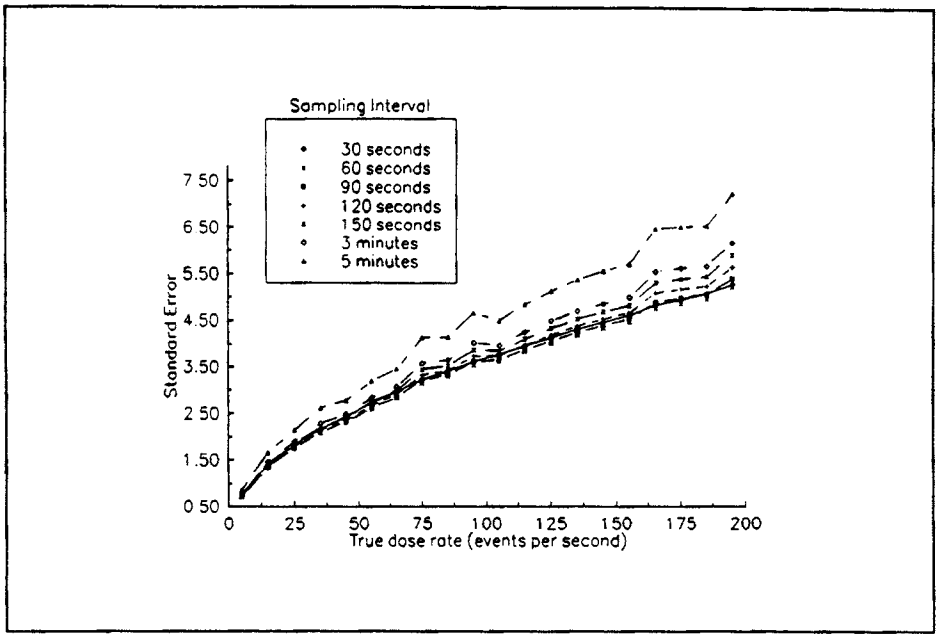

Figure 3. Standard error of estimate plotted against true dose-rate for various sampling intervals.

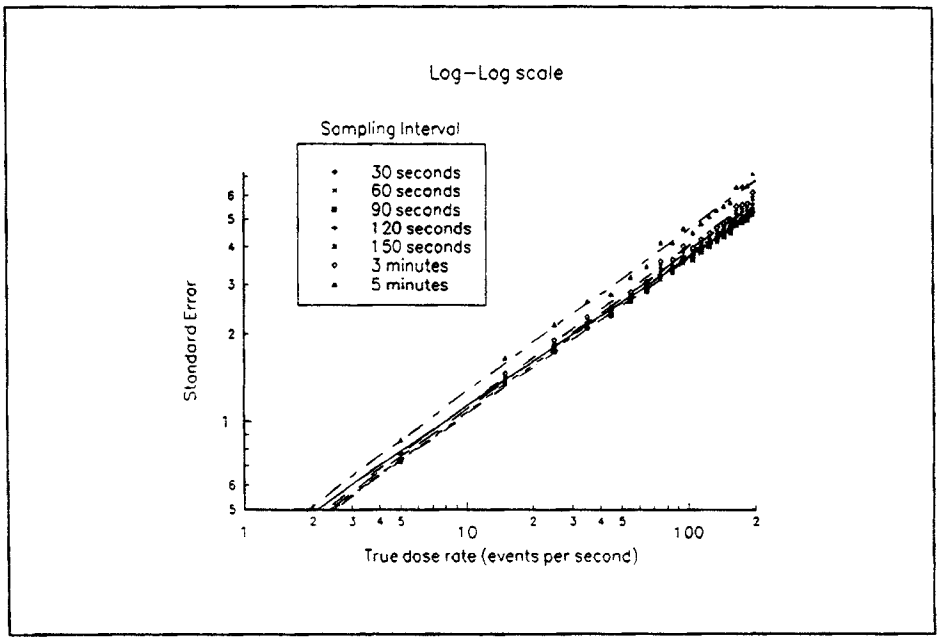

Figure 4. Figure 3 data with standard errors plotted on log-log scale.

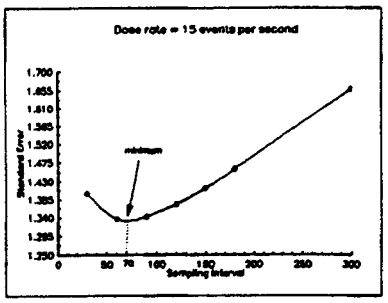

Fig. 5(a)

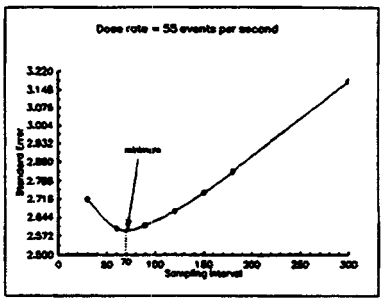

Fig. 5(b)

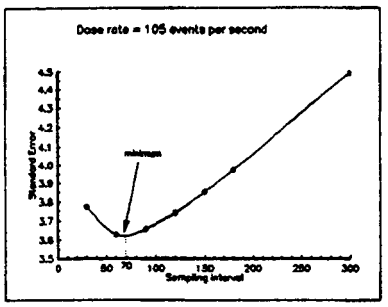

Fig. 5(c)

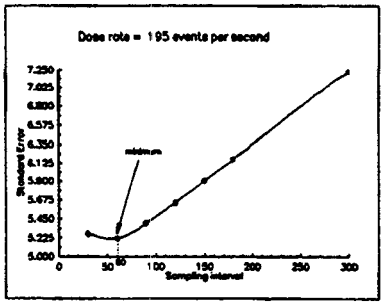

Fig. 5(d)

Figure 5. Relationship between the standard error of $\hat{\lambda}$ and the sampling interval for selected dose-rates.

The sequence of plots in Figure 5(a–d) shows the relationship between the standard error of $\hat{\lambda}$ and the sampling interval for each of the dose-rates considered. Of particular interest is the identification of the sampling interval for which the standard error is a minimum. It is evident from the plots of Figure 5 that this minimum is relatively unaffected by the true mean dose-rate and invariably corresponds with a sampling interval of about 70 seconds (i.e. $70\Delta t$). That the minimum variance is associated with the 70th order statistic *regardless of the* λ *used in Eq. (3.3.2)* is equivalent to saying that this result holds *regardless of the* Δt *used in Eq. (3.3.1)*. On the basis of this observation we conclude that the optimal sampling period is $70\Delta t$ time units where Δt is the time constant of the instrument. What this effectively says is that we only need observe 70 discretizations of the sampling interval and whether this is done once every 1 second, once every two seconds, or at some other sampling rate is irrelevant. Values of α have been determined for the $70\Delta t$ sampling period which provide unbiased estimation of λ. The results are displayed in Table 1.

True Mean Dose-Rate (events/sec)	Coeff. of $Y_{\min}$	Coeff. of $Y_{\max}$
5	0.5789	0.4211
15	0.5424	0.4576
25	0.5324	0.4676
35	0.5272	0.4728
45	0.5241	0.4759
55	0.5217	0.4783
65	0.5199	0.4801
75	0.5180	0.4820
85	0.5172	0.4828
95	0.5160	0.4840
105	0.5155	0.4845
115	0.5147	0.4853
125	0.5141	0.4859
135	0.5135	0.4865
145	0.5130	0.4870
155	0.5126	0.4874
165	0.5119	0.4881
175	0.5116	0.4884
185	0.5114	0.4886
195	0.5109	0.4891

Table 1. Coefficients of $Y_{\min}$ and $Y_{\max}$ for unbiased estimation of true mean dose-rate for sampling interval of $70\Delta t$.

The use of appropriate coefficients from Table 1 will, in most cases, not be practical since the true mean dose-rate is required to be known in advance before the correct weightings for $Y_{\min}$ and $Y_{\max}$ can be assigned. Obviously, if the true mean dose-rate is known then there is no requirement to estimate it. One solution to this 'circular' problem is to obtain a more general calibration equation by estimating the regression equation relating λ to $E(Y_1)$ and $E(Y_k)$. This approach provides *one* coefficient for $Y_{\min}$ and *one* coefficient for $Y_{\max}$ and in a sense, is akin to averaging the coefficients in Table 1 over all dose-rates. Although the procedure eliminates the need to know the true mean dose-rate, there will be a small sacrifice in unbiasedness as only one set of coefficients will be used for all dose-rates. The extent of this bias will depend on the actual value of λ, however it has been calculated that this will typically be less than 1%. Details of the regression approach are given in the following section.

4. Calibration Using Inverse Regression

4.1 SUR Model

We wish to establish a relationship between the true mean dose-rate (λ) and $\mu = E(Y_1)$ and $\mu = E(Y_k)$. It would be natural to think of μ_1 and μ_k as dependent variables and λ as the independent variable, although it will be more useful to perform the inverse regression of λ on μ_1 and μ_k. Note, that the regression analysis is purely a curve-fitting exercise as our data will consist of μ_1 and μ_k values rather than observations on Y_1 and Y_k. Accordingly, no attempt will be made to make any inference based on estimated parameter values.

We commence by letting i index the λ values and j index the sampling intervals ($i = 1,2,\ldots,n$; $j = 1,2,\ldots,k$). For each sampling interval we regress the λ_i data on μ_{1i} and μ_{ki}. Thus, for the jth sampling interval our model is:

$$\Lambda_j = Y_j\beta_j + \epsilon_j \qquad j = 1,2,\ldots,k \tag{4.1.1}$$

where Λ_j is an $(n\times 1)$ vector of dose-rates (λ); Y_j is an $(n\times 2)$ matrix of values of Y_1 and Y_k; β_j is a (2×1) vector of constants; and ϵ_j is an $(n \times 1)$ vector of random errors having zero mean and finite variance $\tilde{\sigma}_j^2$. The k equations implied by Eq. (4.1.1) can be incorporated into a single matrix equation. The following representation is referred to as the Seemingly Unrelated Regression model in econometric literature (Zellner 1962).

$$\Lambda = Y\beta + \epsilon \tag{4.1.2}$$

where Λ is an $(nk \times 1)$ vector equal to $\text{vec}(\Lambda_j)$; Y is an $(nk \times 2k)$ block-diagonal matrix with elements Y_j; β is a $(2k \times 1)$ vector of constants equal to $\text{vec}(\beta_j)$ and ϵ is an $(nk \times 1)$ vector of random error components equal to $\text{vec}(\epsilon_j)$.

For the ith value of λ in the jth sampling interval we have the scalar representation:

$$\Lambda_{ij} = \beta_{1j}(Y_1)_{ij} + \beta_{2j}(Y_k)_{ij} + \epsilon_{ij} \quad . \tag{4.1.3}$$

Now:

$$E(\Lambda_{ij}) = \beta_{1j}(\mu_1)_{ij} + \beta_{2j}(\mu_k)_{ij}$$

and

$$\begin{aligned}\text{Var}(\Lambda_{ij}) &= \beta_{1j}^2\,\text{Var}(Y_1)_{ij} + \beta_{2j}^2\,\text{Var}(Y_k)_{ij} + 2\beta_{1j}\beta_{2j}\,\text{Cov}\{(Y_1)_{ij}(Y_k)_{ij}\} + \tilde{\sigma}_j^2 \\ &= \sigma_{ij}^2 + \tilde{\sigma}_j^2 \quad .\end{aligned}$$

We will assume that $\tilde{\sigma}_j^2 \ll \sigma_{ij}^2$ and thus $\text{Var}(\Lambda_{ij}) \cong \sigma_{ij}^2$.

Now,

$$\text{Var}(\Lambda_{ij}) = \beta_j^T\,\text{Cov}(Y_{ij})\beta_j$$

where Y_{ij} is a (1×2) vector corresponding to the ith row of Y_j.

Next let Ψ be the $(nk \times nk)$ covariance matrix for Λ and Σ_j the $(n \times n)$ covariance matrix corresponding to the jth sampling period, that is

$$\Sigma_j = \text{diag}\{\sigma_{ij}^2\} \qquad i = 1, 2, \ldots, n \ .$$

One structure for Ψ is $\text{diag}\{\Sigma_j\}$; $j = 1, \ldots, k$, that is:

$$\Psi_1 = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & \ddots & \\ & & & \Sigma_k \end{bmatrix} \ . \tag{4.1.5}$$

Since Σ_j is diagonal, Ψ_1 implies observations *within* sampling periods are independent as are observations *between* sampling periods. Independence within sampling periods may be tenable, although may be questionable for between sampling periods as the same dose-rates have been used for each sampling time. Thus, a more general structure for Ψ is:

$$\Psi_2 = \begin{bmatrix} \Sigma_1 & \Gamma_{12} & \Gamma_{13} & \cdots & \Gamma_{1k} \\ \Gamma_{21} & \Sigma_2 & \Gamma_{23} & \cdots & \Gamma_{2k} \\ \vdots & & & & \\ \Gamma_{k1} & \Gamma_{k2} & \Gamma_{k3} & \cdots & \Sigma_k \end{bmatrix} \tag{4.1.6}$$

where Γ_{rs} is an $(n \times n)$ matrix representing the cross-covariance for the n obervations in time period r with those in time period s.

Given a correlation of ρ_{rs} between the λ's of time periods r and s we have:

$$\Gamma_{rs} = \text{diag}(\phi_1, \phi_2, \ldots, \phi_n)$$

where $\phi_m = \rho_{rs}(\Sigma_{rm}\Sigma_{sm})^{1/2}$ with Σ_{rm} the mth diagonal element of Σ_r and Σ_{sm} the mth diagonal element of Σ_s.

4.2 ESTIMATION PROCEDURE: UNRESTRICTED CASE

Parameter estimation is via estimated generalized least squares (EGLS) (Judge, *et al.* 1980, p 117). We let $\Psi_{(0)}$ be either Ψ_1 or Ψ_2 as defined in Eqs. (4.1.5) and (4.1.6) evaluated for the initial vector of parameter estimates:

$$\beta_{(0)} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \otimes 1$$

where 1 is a $(k \times 1)$ vector of ones. The recursive step is:

$$\beta_{(m)} = \left\{ Y^T \Psi_{(m-1)}^{-1} Y \right\}^{-1} Y^T \Psi_{(m-1)}^{-1} \Lambda \tag{4.2.1}$$

where $\Psi_{(m-1)}$ is either Ψ_1 or Ψ_2 evaluated at $\beta_{(m-1)}$. The iterations proceed until $\beta_{(m)}$ and $\beta_{(m-1)}$ differ by less than some prescribed amount in all elements.

4.3 Estimation Procedure: Restricted Case

So that our regression estimators are more consistent with the weightings applied in Eq. (2.2.1), we impose the constraint:

$$1^T \beta_j = 1 \qquad \forall j$$

where 1 is a (2×1) vector of ones. Thus β is estimated subject to:

$$R\beta = r$$

where R is a $(k \times 2k)$ matrix having general form:

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}$$

and r is a $(k \times 1)$ vector of ones.

Denoting the constrained estimator of β at the mth iteration as $\beta^*_{(m)}$, we have

$$\beta^*_{(m)} = \beta_{(m)} + \left(Y^T \Psi^{-1}_{(m-1)} Y\right)^{-1} R^T \left\{ R \left(Y^T \Psi^{-1}_{(m-1)} Y\right)^{-1} R \right\}^{-1} \left(r - R\beta_{(m)}\right) . \tag{4.3.1}$$

Equations (4.2.1) and (4.3.1) have been applied to the data previously analyzed in §3. Results of these calculations are given below.

4.4 Results

The iterations indicated by Eq. (4.3.1) were programmed in the matrix-based language GAUSS and run on a 386-PC computer. Starting with the initial estimate $\beta_{(0)}$ given in §4.2, convergence to within $\pm 10^{-6}$ was typically reached after three or four iterations. For comparison, β has been estimated under a number of different schemes. These results are displayed in Table 2.

As previously mentioned, the aim of the regression analysis is descriptive rather than inferential and thus we make no attempt to declare differences in parameter estimates as "significant" or otherwise. Nevertheless, some general observations may be made from the results presented in Table 2. It would appear that the major difference in parameter estimates is between those obtained using OLS and those estimated using EGLS. A comparison of parameter estimates using Ψ_1 with the corresponding parameters based on Ψ_2 reveal little if no change. We therefore tentatively conclude that the parameter estimates are relatively insensitive to the magnitude and sign of the assumed correlations between sampling intervals.

Time Int.	Param.	Parameter Estimates Using Method: A	B	C	D	E	F	G	H
	β_0	0.5363	0.5126	0.5557	0.5181	0.5553	0.5182	0.5555	0.5183
30	β_1	0.4709	0.4874	0.4576	0.4819	0.4579	0.4818	0.4578	0.4817
	β_0	0.5430	0.5145	0.5670	0.5212	0.5666	0.5212	0.5668	0.5212
60	β_1	0.4666	0.4855	0.4510	0.4788	0.4513	0.4788	0.4512	0.4788
	β_0	0.5465	0.5154	0.5729	0.5226	0.5725	0.5226	0.5728	0.5226
90	β_1	0.4646	0.4846	0.4479	0.4774	0.4481	0.4774	0.4479	0.4774
	β_0	0.5486	0.5159	0.5765	0.5235	0.5763	0.5234	0.5767	0.5234
120	β_1	0.4634	0.4841	0.4461	0.4765	0.4463	0.4766	0.4460	0.4766
	β_0	0.5501	0.5161	0.5789	0.5239	0.5788	0.5239	0.5793	0.5239
150	β_1	0.4626	0.4839	0.4451	0.4761	0.4452	0.4761	0.4481	0.4761
	β_0	0.5512	0.5162	0.5807	0.5242	0.5806	0.5242	0.5812	0.5242
180	β_1	0.4622	0.4838	0.4444	0.4758	0.4444	0.4758	0.4441	0.4758
	β_0	0.5533	0.5161	0.5844	0.5244	0.5846	0.5245	0.5851	0.5245
300	β_1	0.4616	0.4839	0.4434	0.4756	0.4433	0.4755	0.4430	0.4755

Method A is unconstrained OLS;

Method B is constrained OLS;

Method C is uncontrained EGLS with covariance matrix Ψ_1;

Method D is constrained EGLS with covariance matrix Ψ_1;

Method E is unconstrained EGLS with covariance matrix Ψ_2 and associated correlation matrix given below;

Method F is constrained version of method E;

Method G is same as method E but with sign reversal on elements of correlation matrix;

Method H is same as same as method F but with sign reversal on elements of correlation matrix.

Correlation Matrix

Time Interval:	60	90	120	150	180	300
30	0.8199	0.6830	0.5790	0.5000	0.4400	0.3126
60		0.8199	0.6830	0.5790	0.5000	0.3324
90			0.8199	0.6830	0.5790	0.3586
120				0.8199	0.6830	0.3932
150					0.8199	0.4387
180						0.4987

Table 2. Regression coefficients of $Y_{\min}$ (β_0) and $Y_{\max}$ (β_1).

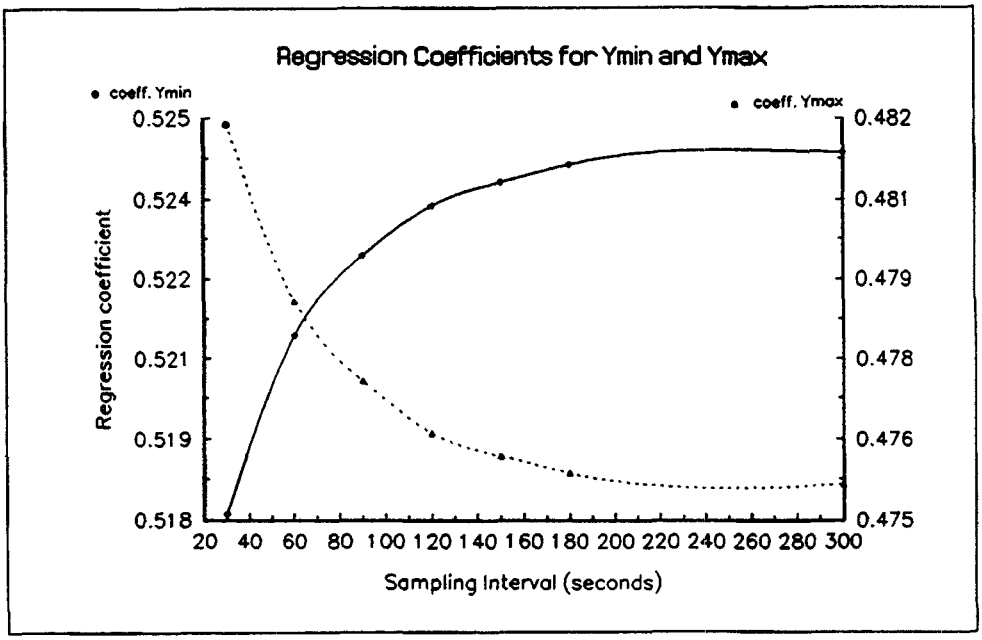

Figure 6. Plot of regression coefficients versus sampling interval.

For future use we have adopted the set of coefficients under column D of Table 2. A plot of these coefficients versus the sampling interval is shown in Figure 6. Interpolating for a sampling interval of 70 seconds using the coefficients in column D of Table 2 we obtain weightings of 0.5217 and 0.4783 for $Y_{\mathbf{min}}$ and $Y_{\mathbf{max}}$ respectively. We note that the average of the coefficients in Table 1 are 0.5214 and 0.4786.

5. An Example

We apply the methods presented in this paper using data obtained from laboratory-based measurements using a hand-held instrument and a certified standard radioactive source. Minimum and maximum meter readings were recorded for a variety of combinations of sampling times and true mean dose-rate. For each $Y_{\mathbf{min}}$ and $Y_{\mathbf{max}}$ pair, three estimates of the mean dose-rate were computed. The first uses the coefficients derived from Figure 2; the second is the simple average of the two extremes; and the third uses the regression coefficients given under column D of Table 2. The resulting mean squared errors are summarized in Table 3.

It is apparent from the results in Table 3 that the regression estimator had smaller MSE in all cases. The large values associated with the true dose-rate of 100 μRad/hr are a function of equipment bias and not of the estimation procedure (the particular instrument used had a propensity to over estimate the true dose-rate).

Sampling Time	30	60	90	120	600	ALL
	61.5	14.4	28.2	47.8	36.6	141.6
25	65.1	14.6	32.5	52.2	39.1	45.0
	61.9	14.4	28.5	48.2	33.0	41.4
	244.3	175.4	107.0	124.5	105.2	151.3
50	249.9	182.0	113.4	133.4	110.2	157.8
	242.7	173.3	105.0	121.4	97.8	148.0
	79.6	216.4	142.6	631.6	184.4	250.9
75	83.6	226.4	151.8	649.0	193.0	260.8
	77.2	210.2	136.8	620.7	171.9	243.4
	1632.3	1431.0	1510.0	1623.8	1432.5	1525.9
100	1675.0	1466.7	1562.5	1679.2	1487.5	1574.2
	1595.8	1399.6	1462.9	1572.7	1352.1	1476.6
	544.7	548.3	506.8	657.7	520.3	556.8
ALL	559.6	564.0	526.8	680.8	541.2	575.8
	533.7	536.4	491.1	640.1	489.8	539.6

Table 3. MSE for estimators 1, 2, and 3 (see text for description).

6. CONCLUSIONS

In this paper we have addressed two fundamental questions associated with radiation dosimetry — namely the determination of an appropriate sampling period and the construction of certain best estimators for the true mean dose-rate. On the basis of the foregoing investigations it is recommended that a sampling interval of $70\Delta t$ be used for radiation studies in which the true mean dose-rate is expected to be less than 200 μRad/hour (Δt is the instrument's time constant). For this sampling interval an approximate minimum variance unbiased estimator of the true mean dose-rate is $(0.5214\,Y_{\mathbf{min}} + 0.4786\,Y_{\mathbf{max}})$ where $Y_{\mathbf{min}}$ and $Y_{\mathbf{max}}$ are, respectively, the smallest and largest meter readings observed during the $70\Delta t$ sampling interval.

ACKNOWLEDGEMENTS

Funding for this work was provided by ALCOA of Australia.

References

Ackers, J.G. (1978), "Accuracy and precision in calibration of low-level radiation monitors". In *National and International Standardization of Radiation Dosimetry*. Proceedings of a symposium, Atlanta December 5–9, 1977. Vienna: International Atomic Energy Agency.

Cember, H. (1983). *Introduction to Health Physics*, 2nd edition. Pergamon Press.

David, H. A. (1981). *Order Statistics*. John Wiley & Sons.

Judge, G.G., W.E. Griffiths, R.C. Hill and L. Tsoung-Chao (1980). *The Theory and Practice of Econometrics*. John Wiley & Sons.

McLaughlin, J.E. (1978), "Instrument calibrations for environmental surveillance". In *National and International Standardization of Radiation Dosimetry*. Proceedings of a symposium, Atlanta December 5–9, 1977. Vienna: International Atomic Energy Agency.

Mann, W.B., R.L. Ayres and S.B. Garfinkel (1980). *Radioactivity and its Measurement*. Pergamon Press.

NCRP (1978), "Instrumentation and monitoring methods for radiation". National Council on Radiation Protection and Measurements Report No. 57.

NCRP (1985a), "Environmental measurements", first reprinting. National Council on Radiation Protection and Measurements, Report No. 50.

NCRP (1985b), "A handbook of radioactivity measurements and procedures", Second edition. National Council on Radiation Protection and Measurements Report No. 58.

Tsoulfanidis, N. (1983). *Measurement and Detection of Radiation*. McGraw-Hill.

Zellner, A. (1962), "An efficient method of estimating seemingly unrelated regressions and tests of aggregation bias". *Journal of the American Statistical Association* 57, 348–368.