A Statistical Method for Assessing Compliance with Nutrient Reduction Targets

Technical Note

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1. Problem description

In order to set measurable environmental goals, regulatory agencies are increasingly defining environmental 'improvement' relative to some baseline condition. For example, phosphorous loads to the Gippsland Lakes are required to be 40% of their 'current' levels by the year 2005. The baseline load for this determination has been estimated using 1995-97 historical data. The setting of targets in this manner automatically leads us to consider *ratios* of random variables. From a statistical perspective, the analysis of ratios of random variables has difficulties – for example non-existent moments. It is not possible to derive generic results that will allow us to make inference about a target which is defined in terms of a ratio of some environmental parameter measured at different places or times. To do so would require detailed information about the distributional properties of the particular parameter(s) of interest. While general observations are possible for water quality variables (eg. non-negative, positive skewed) it would be unwise to assume a particular distributional form (eg. log-normal) since violations in practice would not be uncommon. An alternative approach has been adopted in this technical note which utilises some approximate results for the ratio of random variables which hold true irrespective of the underlying distribution of values from which the data have been sampled.

2. Mathematical Considerations

Let the random variable of interest be denoted Y (eg. total nutrient load to a system). Subscripts on Y will be used to differentiate between different epochs. Thus Y_B will denote a baseline or reference value and Y_T the corresponding value at some future time, T.

In deciding whether or not a particular reduction target has been met, we need to consider the ratio $\tau = \frac{Y_T}{Y_B}$. Note, that both the numerator and denominator of this expression are treated as random variables[1]. We denote by $\hat{\tau}$ an estimator of τ where $\hat{\tau} = \frac{y_T}{y_B}$ and lowercase y's indicate sample estimates of the corresponding true parameter.

Using a first-order Taylor series expansion it is possible to derive the following approximations:

[1] An alternative and simpler approach is to treat Y_B as fixed or measured without error. This may be appropriate where Y_B cannot be estimated from past data and/or the regulatory agency establishes Y_B by decree – eg. as an 'aspirational' target.

$$\mathrm{E}\left[\hat{\tau}\right] \simeq \frac{Y_T}{Y_B} - \frac{1}{Y_B^2} Cov\left[Y_T, Y_B\right] + \frac{Y_T}{Y_B^3} Var\left[Y_B\right] \tag{2.1}$$

$$Var\left[\hat{\tau}\right] \simeq \left(\frac{Y_T}{Y_B}\right)^2 \left\{ \frac{Var\left[Y_T\right]}{Y_T^2} + \frac{Var\left[Y_B\right]}{Y_B^2} - \frac{2Cov\left[Y_T, Y_B\right]}{Y_T Y_B} \right\} \tag{2.2}$$

Equations (2.1) and (2.2) can be simplified if it can be assumed that Y_T and Y_B are *independent*. In this case we have:

$$\mathrm{E}\left[\hat{\tau}\right] \simeq \frac{Y_T}{Y_B} + \frac{Y_T}{Y_B^3} Var\left[Y_B\right] \tag{2.3}$$

and

$$Var\left[\hat{\tau}\right] \simeq \left(\frac{Y_T}{Y_B}\right)^2 \left\{ \frac{Var\left[Y_T\right]}{Y_T^2} + \frac{Var\left[Y_B\right]}{Y_B^2} \right\} \tag{2.4}$$

Equation (2.3) can be rewritten as

$$\mathrm{E}\left[\hat{\tau}\right] \simeq \tau\left(1 + CV_B^2\right) \tag{2.5}$$

where CV_B is the *coefficient of variation* (defined here as the standard deviation relative to *Y*) of the baseline value. It is evident from equation (2.5) that the approximation *overestimates* the true ratio. Furthermore, the quantity CV_B is unlikely to be known. To overcome this we consider the related quantity $\tau' = 1/\tau = \frac{Y_B}{Y_T}$.[2]

We thus obtain:

$$\mathrm{E}\left[\hat{\tau}'\right] \simeq \tau\left(1 + CV_T^2\right) \tag{2.6}$$

and

$$Var\left[\hat{\tau}'\right] \simeq \tau'^2\left(CV_B^2 + CV_T^2\right) \tag{2.7}$$

[2] Note that if we were looking for a 40% reduction say, then τ would be equal to 0.6 and thus τ' would be equal to 1/0.6 = 1.67.

3. Hypothesis Testing

The assessment of whether or not a target has been attained can be conducted within a conventional hypothesis testing framework[3]. Thus, we are interested in testing:

$$H_0: \quad \tau' = \theta_0$$

versus

$$H_1: \quad \tau' = \theta_1 \qquad \text{where } \theta_1 > \theta_0 .$$

For example, a test of a minimum 40% reduction against a null hypothesis of no change is equivalent to:

$$H_0: \quad \tau' = 1$$
$$H_1: \quad \tau' \geq 1.67$$

Define the test statistic: $Z = \dfrac{\hat{\tau}' - \tau'(1+CV_T^2)}{\sqrt{Var\left[\hat{\tau}'\right]}}$ and define a critical value, τ^* for an α-level test as :

$$\tau^* = \theta_0\left(1+CV_T^2\right) + z_{1-\alpha}\sqrt{Var\left[\hat{\tau}'\right]} \qquad (2.8)$$

Now, the *power* of the test is defined as:

$$P\left[\hat{\tau}' > \tau^* \middle| \tau' = \theta_1\right] = 1-\beta \qquad (2.9)$$

where β is the Type II error rate.

From equation (2.9) we obtain

$$\frac{\theta_0\left(1+CV_T^2\right) + z_{1-\alpha}\sqrt{Var\left[\hat{\tau}'\right]} - \theta_1\left(1+CV_T^2\right)}{\sqrt{Var\left[\hat{\tau}'\right]}} = z_\beta \qquad (2.10)$$

and by substituting equation (2.7) into equation (2.10)

$$\theta_1 CV_T\left(\frac{CV_B^2}{CV_T^2}+1\right)^{1/2} = \frac{\left(1-\theta_1\right)\left(1+CV_T^2\right)}{z_\beta - z_{1-\alpha}} \qquad (2.10)$$

Equation (2.11) can be simplified if it can be assumed that the relative variation (coefficient of variation) in the baseline and future estimates are equal. Proceeding on this assumption we have

$$\left(1-\theta_1\right)CV_T^2 - CV_T\left[\sqrt{2}\,\theta_1\left(z_\beta - z_{1-\alpha}\right)\right] + \left(1-\theta_1\right) = 0 \qquad (2.11)$$

[3] The technical development of the hypothesis test is predicated on the assumption that the test statistic is normally distributed. To a first approximation this is not unreasonable and has been shown to hold in preliminary assessments.

Equation (2.12) defines a quadratic in CV_T which has solutions:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (2.12)$$

with

$$a = (1 - \theta_1)$$

$$b = -\sqrt{2}\,\theta_1 \left(z_\beta - z_{1-\alpha}\right)$$

and

$$c = b = (1 - \theta_1)$$

Generally, there will only be one solution to (2.12) which makes sense (eg. second solution is negative).

Equation (2.12) can be used to obtain plots of the coefficient of variation against power for various levels of significance, α. Examples of such plots are provided below.

Example

Suppose we are interested in establishing whether or not a 40% nutrient load reduction has been achieved using a test with a 5% level of significance. Furthermore, we require that the statistical test has power of at least 80% (ie. the test will correctly reject the null hypothesis of no difference with 0.8 probability when in fact the target has been attained).

By referring to the blue curve of Figure 3, it can be seen that this requirement will be met provided the coefficient of variation for the load estimate is no more than about 0.13 (ie. the standard deviation of the estimated load is no bigger than 13% of the load itself).

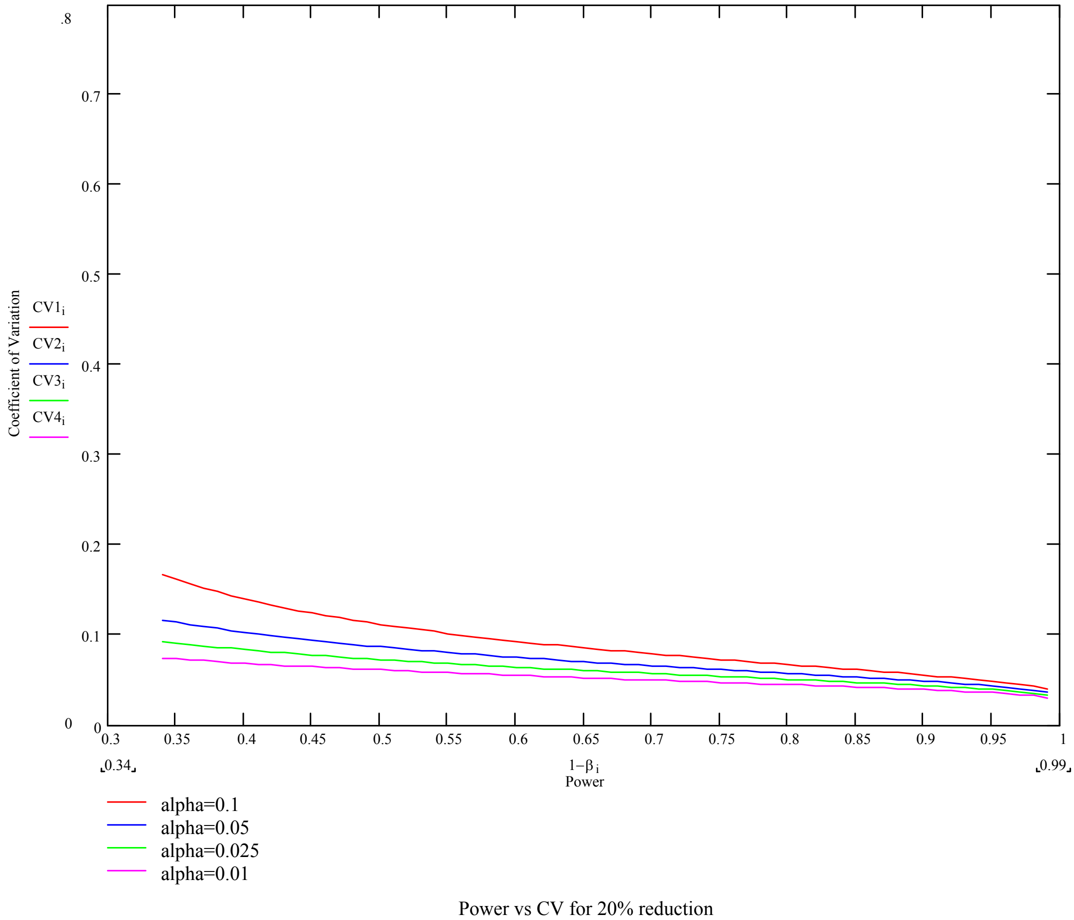

Figure 1 CV – Power curves for testing a hypothesis of no change against an alternative hypothesis of a 20% reduction.

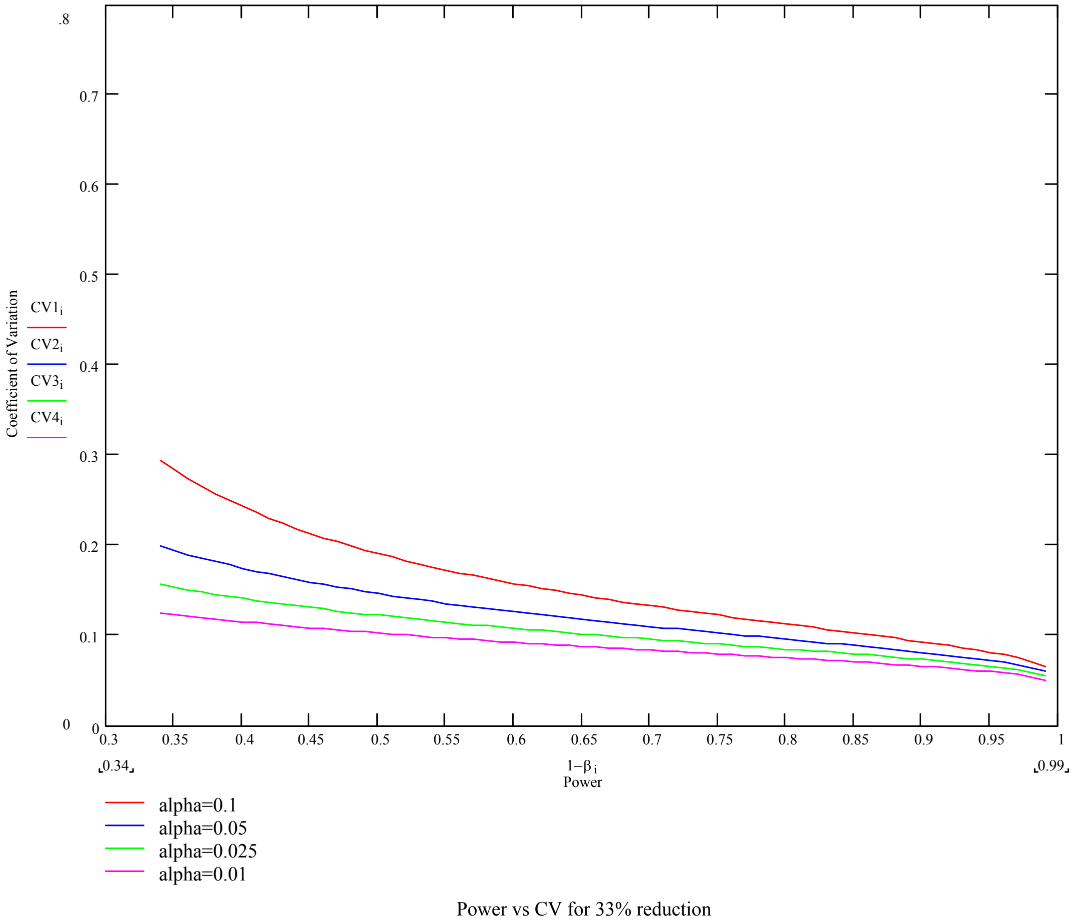

Figure 2 CV – Power curves for testing a hypothesis of no change against an alternative hypothesis of a 33% reduction.

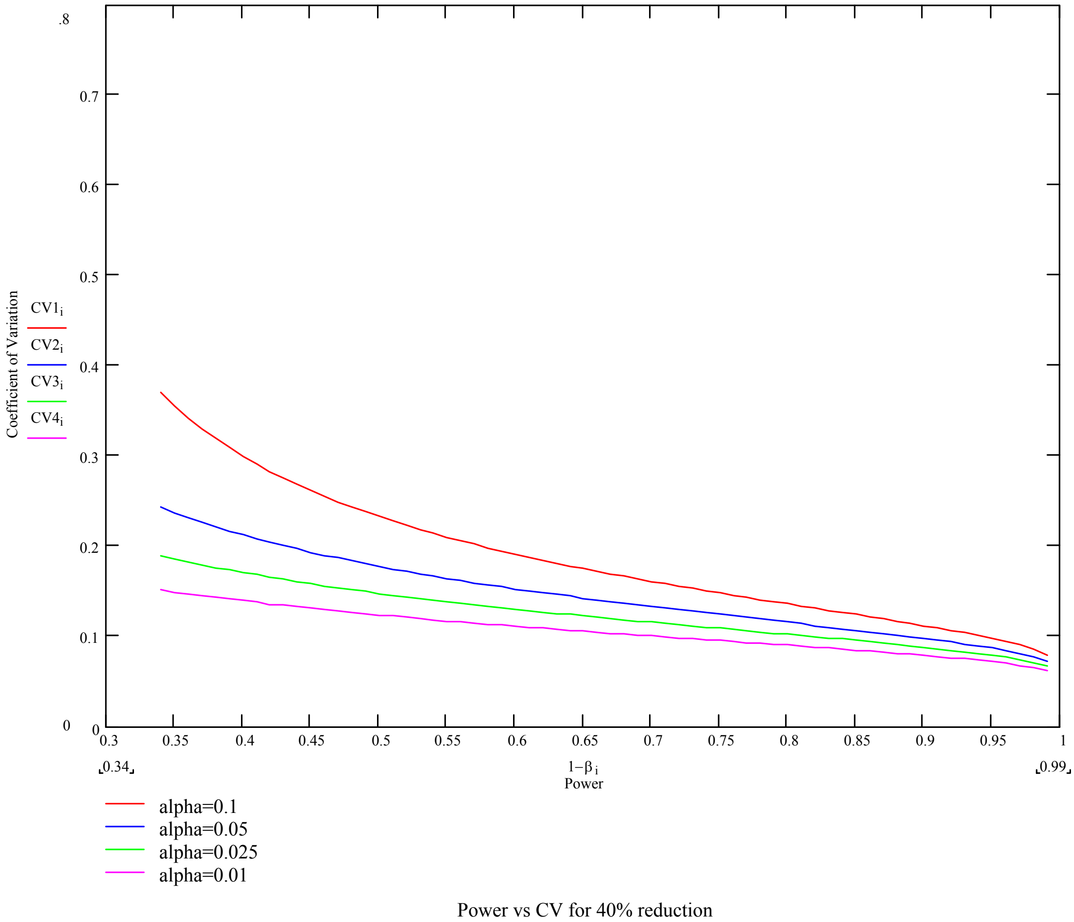

Figure 3 **CV – Power curves for testing a hypothesis of no change against an alternative hypothesis of a 40% reduction.**

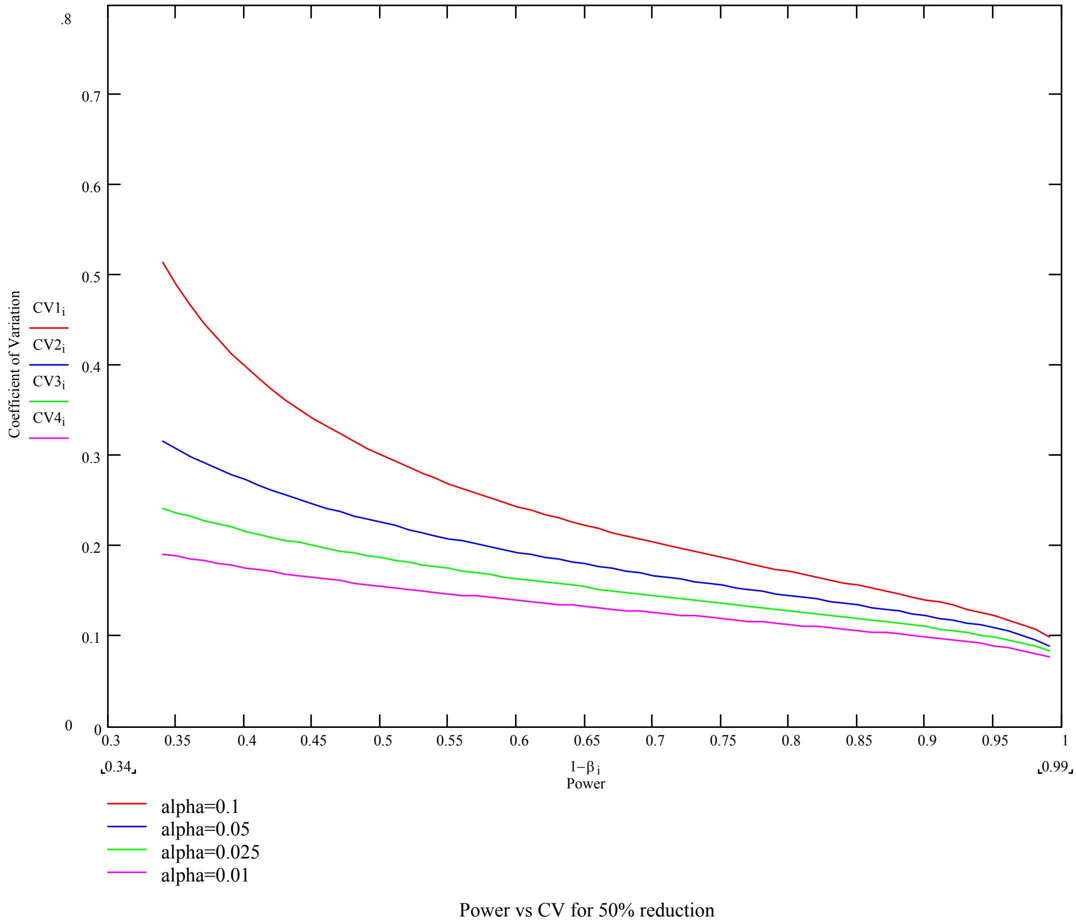

Figure 4 CV – Power curves for testing a hypothesis of no change against an alternative hypothesis of a 50% reduction.

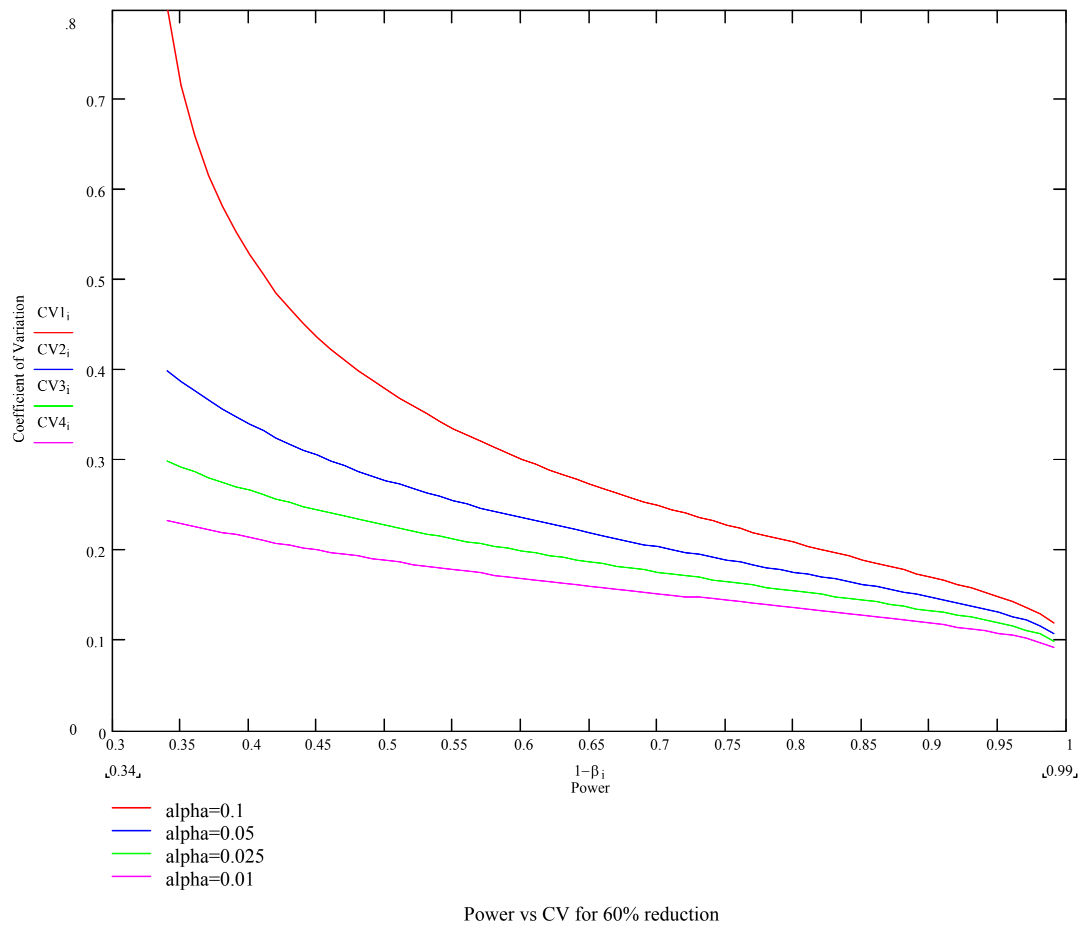

Figure 5 CV – Power curves for testing a hypothesis of no change against an alternative hypothesis of a 60% reduction.

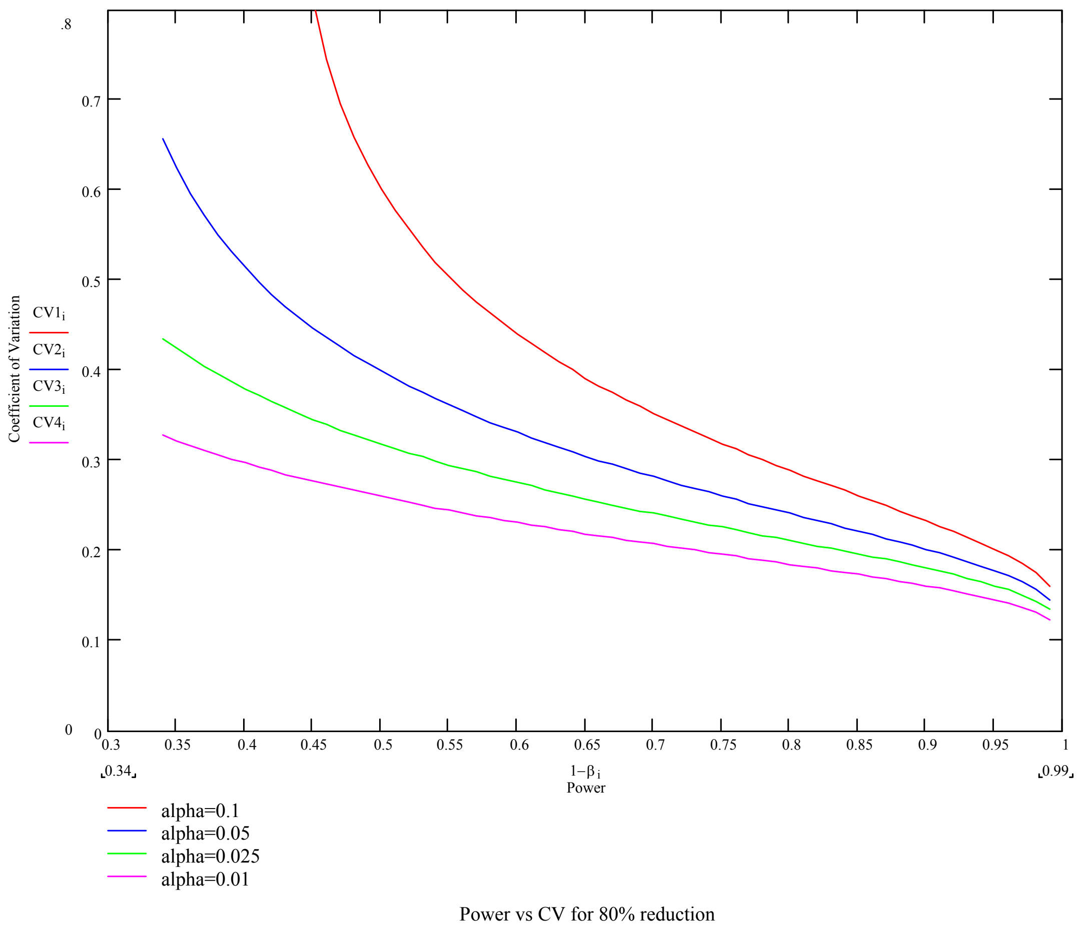

Figure 6 CV – Power curves for testing a hypothesis of no change against an alternative hypothesis of a 80% reduction.